Purposeful Academic Classes for Excelling Students Program

Mathematics Methods Units 3 & 4

Session 1

Exponential functions

3.1.1 estimate the limit of $\frac{a^h-1}{h}$ as $h \to 0$, using technology, for various values of $a > 0$

3.1.2 identify that e is the unique number a for which the above limit is 1

3.1.3 establish and use the formula $\frac{d}{dx}(e^x) = e^x$

3.1.4 use exponential functions of the form Ae^{kx} and their derivatives to solve practical problems

Trigonometric functions

3.1.5 establish the formulas $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ by graphical treatment, numerical estimations of the limits, and informal proofs based on geometric constructions

3.1.6 use trigonometric functions and their derivatives to solve practical problems

Differentiation rules

3.1.7 examine and use the product and quotient rules

3.1.8 examine the notion of composition of functions and use the chain rule for determining the derivatives of composite functions

3.1.9 apply the product, quotient and chain rule to differentiate functions such as xe^x, $\tan x$, $\frac{1}{x^n}$, $x \sin x$, $e^{-x} \sin x$ and $f(ax - b)$

3.1.10 use the increments formula: $\delta y \approx \frac{dy}{dx} \times \delta x$ to estimate the change in the dependent variable y resulting from changes in the independent variable x

3.1.11 apply the concept of the second derivative as the rate of change of the first derivative function

3.1.13 examine the concepts of concavity and points of inflection and their relationship with the second derivative

3.1.14 apply the second derivative test for determining local maxima and minima

3.1.15 sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection

3.1.16 solve optimisation problems from a wide variety of fields using first and second derivatives

Logarithmic functions

4.1.1 define logarithms as indices: $a^x = b$ is equivalent to $x = \log_a b$ i.e. $a^{\log_a b} = b$

4.1.2 establish and use the algebraic properties of logarithms

4.1.3 examine the inverse relationship between logarithms and exponentials: $y = a^x$ is equivalent to $x = \log_a y$

4.1.4 interpret and use logarithmic scales

4.1.5 solve equations involving indices using logarithms

4.1.6 identify the qualitative features of the graph of $y = \log_a x$ $(a > 1)$, including asymptotes, and of its translations $y = \log_a x + b$ and $y = \log_a(x - c)$

4.1.7 solve simple equations involving logarithmic functions algebraically and graphically

4.1.8 identify contexts suitable for modelling by logarithmic functions and use them to solve practical problems

Calculus of the natural logarithmic function

4.1.9 define the natural logarithm $\ln x = \log_e x$

4.1.10 examine and use the inverse relationship of the functions $y = e^x$ and $y = \ln x$

4.1.11 establish and use the formula $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Euler's Number *e*

- Euler's number, e, is a special type of irrational number and is called a transcendental number and rounded to 30 decimal places is 2.718 281 828 459 045 235 360 287 471 357

- $\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t = e \iff \lim_{x\to\infty}\sum_{n=0}^{x}\left(\frac{1}{n!}\right) = e \iff \lim_{t\to 0}(1+t)^{\frac{1}{t}} = e$

- The expression $\lim_{t\to 0}\left(\frac{a^t - 1}{t}\right) = 1$ as $a \to e$.

Example 1 Calculator Assumed

The number (N) of bats in a given region is modelled by $N = 10\,000\ e^{0.008t}$ where t is the number of years after January 2015.

(a) Calculate the percentage rate of increase in the bat population in January 2020.

(b) Calculate the average percentage rate of increase in the bat population between January 2015 and January 2020.

(c) Explain the difference in the answers in (a) and (b).

(d) The time it takes for the population to double is called the doubling time. Calculate the doubling time for the bats in this region.
Give your answer to the nearest year.

(e) When to the nearest year would the number of bats double from the number of bats in January 2020? Justify your answer.

In January 2020, a wind farm built in the region became operational. The number of bats t years after January 2020 is now modelled by the equation $P = P_0 e^{0.001t}$.

(f) State the value of P_0 and calculate the expected rate of increase in the number of bats in January 2025.

(g) Describe with reasons how the wind farm has affected the number of bats in this region.

(2 marks)

Example 2 Calculator Assumed

The mass of chemical A in the blood stream of a patient t hours after 8 am is given by $M = 100\ e^{-kt}$ mg. Within each 4 hour interval, 40% of the chemical present at the start of the 4 hour interval, is absorbed by the patient's body and disappears from the blood stream.

(a) Find k to four significant figures.

(b) For the chemical to be effective, there must be at least 36 mg of the chemical in the blood stream of the patient at all times. This can be achieved if the patient is given a dose of 100 mg of the chemical every h hours. Calculate the value of h.

(c) Calculate the rate of absorption of the chemical by the patient's body at 12 noon.

Logarithms

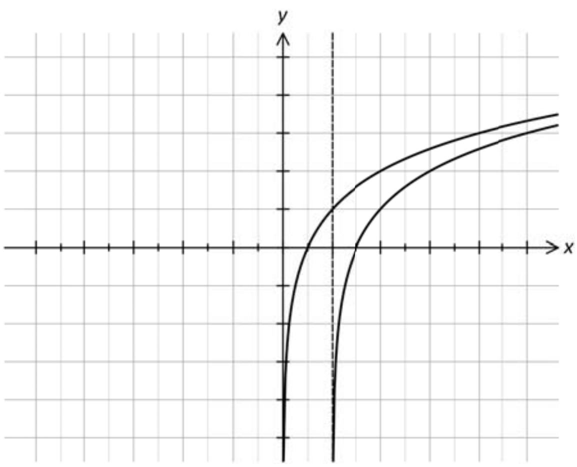
$x = \log_a b \Leftrightarrow a^x = b$	$a^{\log_a b} = b$ and $\log_a(a^b) = b$
$\log_a mn = \log_a m + \log_a n$	$\log_a \frac{m}{n} = \log_a m - \log_a n$
$\log_a(m^k) = k\log_a m$	$\log_e x = \ln x$

- Logarithms of one base are related to logarithms of another base through the following formula:

$$\log_a M = \frac{\log_b M}{\log_b a}$$

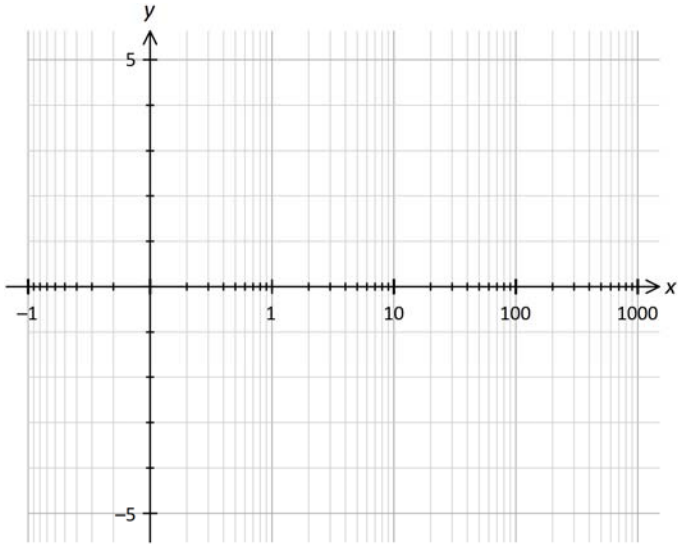
Example 3 Calculator Free

The diagram below shows the graphs of $y = \log_b x$ and $y = \log_b(x - c)$ where b and c are constants. Sketch on the same axis the graph of $y = \log_b\left[x(x-c)\right]$. Show clearly the asymptotes, the x-intercept and one other point.



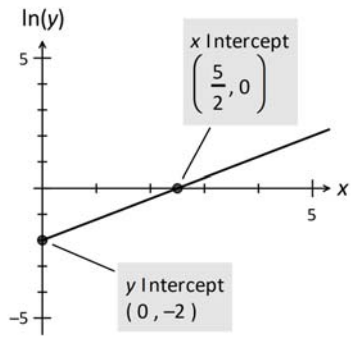
Example 4 **Calculator Free**

The x-axis in the diagram below is drawn using a logarithmic scale.
On the axes below, plot the curve with equation $y = \log(0.1x)$ for $x \geq 1$.



Example 5 **Calculator Free**

The graph of $\ln y$ against x is shown in the accompanying diagram. Determine the algebraic relationship between y and x.



Example 6 **Calculator Free**

Given that $\ln\left(\frac{y}{10-y}\right) = 2t + 1$, show that $y = \frac{a}{Ae^{-kt}+1}$ where A is a constant.

Example 7 Calculator Free

(a) For $x > 0$, let $2^{\ln x} = p$. Show that $x^{\ln 2} = p$.

(b) Solve for x in the equation, $5x^{\ln 2} - 2^{\ln x} = 8$.

Example 8 Calculator Free

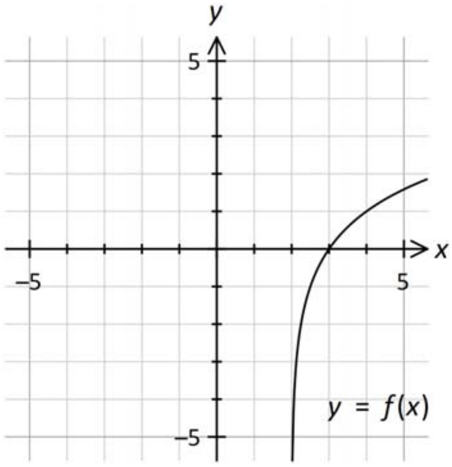
(a) Show that $\log_8 x = \frac{1}{3}\log_2 x$.

(b) Hence, or otherwise solve the equation $\log_2 x + \log_8 x = 8$.

Example 9 Calculator Free

The diagram below shows the graph of the logarithmic function $y = f(x)$.
The function $g(x)$ is the inverse of $f(x)$.

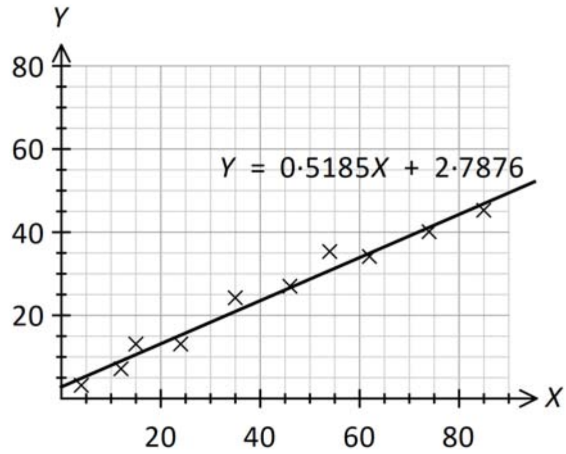
(a) On the same axes, draw the graph of $y = g(x)$.
Identify and label all essential features of the graph of $y = g(x)$.



(b) The logarithmic function $f(x)$ has the form $f(x) = \log_b(x-k)$ where b and k are real constants. The graph of $y = f(x)$ intersects the x-axis at (3, 0) and passes through the point with coordinates (4, 1). Determine with reasons $g(x)$.

Example 10 Calculator Assumed

The diagram below shows a scatter graph of some experimental values.
The line of best fit through these points has equation Y = 0.5185X + 2.7876.



(a) Given that $Y = \log Q$ and $X = t$, use the line of best fit to determine the algebraic relationship between Q and t in the form $Q = A \times 10^{kt}$.
Give A to the nearest whole number and k to two decimal places.

(b) Given that $Y = \ln P$ and $X = \ln t$, where $t > 0$, use the line of best fit and the rules of logarithms to show that $P = 16.24 \times t^{0.52}$.

Differentiation

- The derivative of $f(x)$, denoted $f'(x)$ or $\frac{d}{dx}f(x)$, is defined as:

$$f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$\frac{d}{dx}(x^n) = nx^{n-1}$
$\frac{d}{dx}(e^{ax-b}) = ae^{ax-b}$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$
$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$
$\frac{d}{dx}(\sin(ax-b)) = a\cos(ax-b)$
$\frac{d}{dx}(\cos(ax-b)) = -a\sin(ax-b)$

Product rule	If $y = uv$ then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$	or	If $y = f(x)\,g(x)$ then $y' = f'(x)\,g(x) + f(x)\,g'(x)$
Quotient rule	If $y = \frac{u}{v}$ then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	or	If $y = \frac{f(x)}{g(x)}$ then $y' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$
Chain rule	If $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	or	If $y = f(g(x))$ then $y' = f'(g(x))\,g'(x)$

Trigonometry

$\sin^2 x + \cos^2 x = 1$	$\tan x = \frac{\sin x}{\cos x}$

Example 11 Calculator Assumed

A radioactive substance undergoes exponential decay with a half-life of 12 years. A_0 is the initial mass of the radioactive substance and $A(t)$ is the mass of the substance (g) left after t hours and.

(a) What proportion of the initial amount of the radioactive substance is left after 48 years?

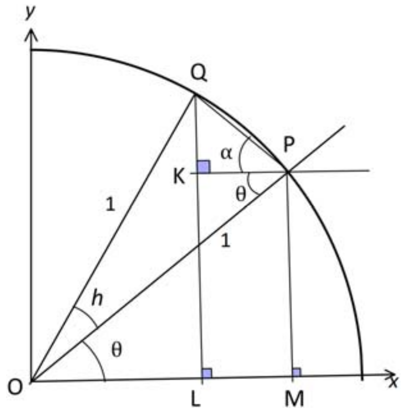
(b) Justify the exponential decay equation $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{12}}$ describing this radioactive substance.

(c) Show that the exponential decay equation $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{12}}$ may be expressed as $A = A_0e^{-kt}$, stating the value of k to four significant figures.

(d) Use differentiation to determine the initial rate of change of the mass of this radioactive substance in terms of A_0 .

(e) Determine the ratio of the rate of change of mass of the radioactive substance after 48 years to the initial rate of change of mass.

Example 12 **Calculator Free**



The points P and Q lie on a unit circle centred at the origin O of the *x*-*y* axes. The lines QKL and PM are parallel to the *y*-axis and the line KP is parallel to the *x*-axis. $\angle POM = \theta$ radians, $\angle POQ = h$ radians and $\angle QPK = \alpha$ radians. Let the length of the chord QP be *u*.

(a) Use triangle POM to find PM in terms of θ.

(b) Use triangle QOL to explain why QL = $\sin(\theta + h)$.

(c) Use your answers in (a) and (b) to determine an expression for QK in terms of θ.

(d) Use triangle QKP to find an expression for QK in terms of α and *u*.

(e) Use the definition, $\dfrac{d}{d\theta}(\sin\theta) = \lim\limits_{h\to 0}\left(\dfrac{\sin(\theta+h)-\sin(\theta)}{h}\right)$,

to show that $\dfrac{d}{d\theta}(\sin\theta) = \lim\limits_{h\to 0}\left(\dfrac{u}{h}\times\sin(\alpha)\right)$.

(f) Explain why the length of the arc QP = *h*.

(g) Explain why as $h \to 0$, the ratio $\dfrac{u}{h} \to 1$.

(h) Explain why as $h \to 0$, the angle $\alpha \to \dfrac{\pi}{2} - \theta$.

(i) Hence, show that $\dfrac{d}{d\theta}(\sin\theta) = \cos\theta$

Example 13 Calculator Free

(a) Use the relationship $e^{ln(b)} \equiv b$, to express 5^x in terms of the Euler number e.

(b) Hence, or otherwise, differentiate 5^x with respect to x.

Stationary Points and Inflection Points

1. Turning points and horizontal inflection points are collectively termed *stationary points.*
2. *When $f'(a) = 0$, then the curve $y = f(x)$ has a stationary point at $x = a$.*
   To identify the nature of the stationary point, one of two tests can be used.
   - The second derivative test:
     - The stationary point is a *maximum turning point* if $f''(a) < 0$,
     - The stationary point is a *minimum turning point* if $f''(a) > 0$,
     - The stationary point is a *horizontal inflection point* if $f''(a) = 0$ and $f''(a^+)$ and $f''(a^-)$ have opposite signs.
   - The sign test (use this test when $f'(x)$ is difficult to obtain):
     - The stationary point at $x = a$, is a maximum turning point if :

x	$x = a^-$	$x = a$	$x = a^+$
sign for dy/dx or $f'(x)$	+ /	0 —	– \

     - The stationary point at $x = a$, is a minimum turning point if :

x	$x = a^-$	$x = a$	$x = a^+$
sign for dy/dx or $f'(x)$	– \	0 —	+ /

     - The stationary point at $x = a$, is a *horizontal inflection point* if :

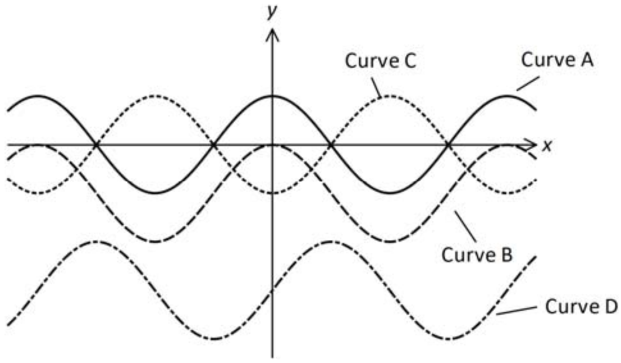
x	$x = a^-$	$x = a$	$x = a^+$
sign for dy/dx or $f'(x)$	– \ **or +** /	0 — 0 —	– \ + /

a^- is a value of x slightly less than a and a^+ is a value of x slightly greater than a.

3. The curve $y = f(x)$ has an inflection point at $x = a$, when $f''(a) = 0$ and $f''(a^+)$ and $f''(a^-)$ have opposite signs.
   If $f'(a) = 0$ at the same time, then the inflection point is a horizontal inflection point.
   If $f'(a) \neq 0$ at the same time, then the inflection point is an oblique inflection point.

Example 14 Calculator Free

(a) The diagram below shows the graphs of four functions. Determine with reasons which curve is the graph of $y = f(x)$ and which curve is the graph of $y = f'(x)$.



(b) Consider the curve with equation $y = (1+\sin x)^2$ for $0 \le x \le \pi$.

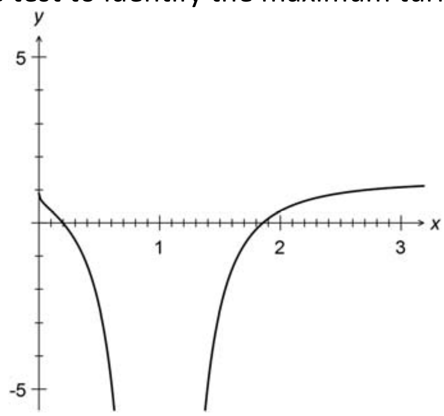
Determine the coordinates of the two inflection points on this curve.

Example 15 Calculator Free

Consider the curve with equation $y = x + \dfrac{x}{\ln x} - 10$.

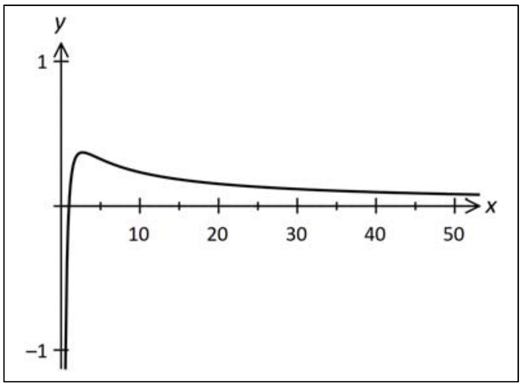
(a) Use an analytical method to show that this curve has turning points when $(\ln x)^2 + \ln x - 1 = 0$.

(b) Use the graph of $y = 1 + \dfrac{\ln x - 1}{(\ln x)^2}$ shown below, to determine correct to one decimal place, the x-coordinate of the maximum point of this curve. Use either the sign test or the second derivative test to identify the maximum turning point.



Example 16 Calculator Free

The accompanying diagram shows the graph of $y = \frac{\ln(x)}{x}$. The graph has a horizontal asymptote with equation $y = 0$.



(a) Determine the coordinates of the maximum point of this function.

(b) Consider the function $y = x^{\left(\frac{1}{x}\right)}$, for $x > 0$. By rewriting the function as $y = e^{\ln x^{\left(\frac{1}{x}\right)}}$:

(i) Show that $\lim_{x \to \infty} x^{\left(\frac{1}{x}\right)} = 1$.

(ii) Determine the coordinates of the maximum point of $y = x^{\left(\frac{1}{x}\right)}$, for $x > 0$.

Example 17 Calculator Free

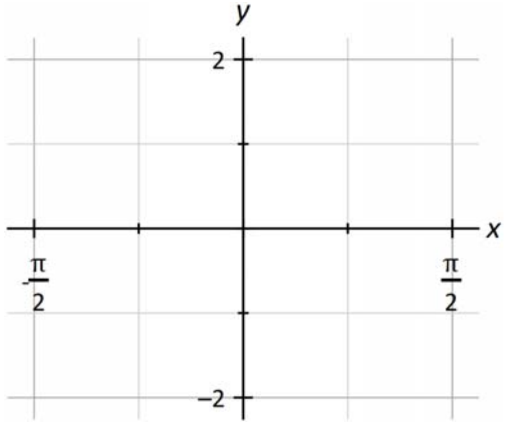
Consider the curve with equation $y = e^{-x}\cos x$ for $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$.

(a) Determine the x-coordinate of the stationary point of this curve.

(b) Use the second derivative test to determine the nature of the stationary point in (a).

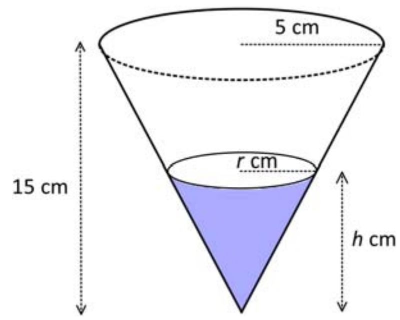
(c) Determine the coordinates of the inflection point on this curve.

(d) On the axes provided below, sketch $y = e^{-x}\cos x$ for $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$.



Example 18 Calculator Assumed

A conical vessel is being filled with water. The open end of the conical vessel is a circle of radius 5 cm. The height of the vessel is 15 cm. The water depth measured from the vertex of the vessel is *h* cm and the radius of the water surface is *r* cm.



Use the incremental formula to find the percentage change in *V*, the volume of water in the vessel corresponding to a 1% increase in the depth of the water level.

Example 19 Calculator Assumed

The radius R (metres) of a circular patch of oil at time t *days* is given by $R = 5e^{0.05t}$.

(a) Use the incremental method to determine the approximate change in the radius of the oil patch at the end of the first *hour*.

(b) Use your answer in (a) to estimate the area of the oil patch at the end of the first hour.

(c) Let A be the area of the circular patch of oil at time t days.

(i) Determine an expression for A in terms of R.
Hence, use the chain rule to determine an expression for the rate of change of A with respect to t. Give your answer in terms of t.

(ii) Determine the rate of change of the area of the patch at the end of the first hour.